A GENERALIZATION OF THE CASSELS-TATE DUAL EXACT SEQUENCE

CRISTIAN D. GONZÁLEZ-AVILÉS AND KI-SENG TAN

Abstract. We extend the well-known Cassels-Tate dual exact sequence for abelian varieties A over global fields K in two directions: we treat the p-primary component in the function field case, where p is the characteristic of K, and we dispense with the hypothesis that the Tate-Shafarevich group of A is finite.

1. Introduction

Let K be a global field and let m be a positive integer *which is prime to the characteristic of* K (in the function field case). Let A be an abelian variety over K. Then there exists an exact sequence of discrete groups

$$0 \to Ш(A)(m) \to H^1(K,A)(m) \to \bigoplus_{\text{all } v} H^1(K_v,A)(m) \to Б(A)(m) \to 0,$$

where K_v is the henselization of K at v, $M(m)$ denotes the m-primary component of a torsion abelian group M, and $Б(A)$ is defined to be the cokernel of the localization map $H^1(K,A) \to \bigoplus_{\text{all } v} H^1(K_v,A)$. The Pontrjagyn dual of the preceding exact sequence is an exact sequence of compact groups

$$0 \leftarrow Ш(A)(m)^* \leftarrow H^1(K,A)(m)^* \leftarrow \prod_{\text{all } v} H^0(K_v,A^t)^{\wedge} \leftarrow Б(A)(m)^* \leftarrow 0,$$

where A^t is the abelian variety dual to A and, for any abelian group M, $M^{\wedge}$ denotes the m-adic completion $\varprojlim_n M/m^n$ of M. Now, if $Ш(A)(m)$ is *finite* (or, more generally, if $Ш(A)(m)$ contains no nontrivial elements which are divisible by m^n for every $n \geq 1$), then $Ш(A)(m)^*$ and $Б(A)(m)^*$ are canonically isomorphic to $Ш(A^t)(m)$

2000 *Mathematics Subject Classification.* Primary 11G35; Secondary 14G25.

Key words and phrases. Selmer groups, Tate-Shafarevich groups, Cassels-Tate dual exact sequence.

C.G.-A. is partially supported by Fondecyt grant 1061209 and Universidad Andrés Bello grant DI-29-05/R.

K.-S.T. is partially supported by the National Science Council of Taiwan, NSC91-2115-M-002-001, NSC94-2115-M-002-010.

and $A^t(K)^{\frown}$, respectively, and the preceding exact sequence induces an exact sequence

$$0 \leftarrow Ш(A^t)(m) \leftarrow H^1(K,A)(m)^* \leftarrow \prod_{\text{all } v} H^0(K_v,A^t)^{\frown} \leftarrow A^t(K)^{\frown} \leftarrow 0$$

which is known as *the Cassels-Tate dual exact sequence* [3, 11]. See [9, Theorem II.5.6(b), p.247]. The aim of this paper is to extend the isomorphism $Б(A)(m)^* \simeq A^t(K)^{\frown}$ recalled above to the case where m is divisible by the characteristic of K (in the function field case) and no hypotheses are made on $Ш(A)$. The following is the main result of the paper. Let m and n be *arbitrary* positive integers. Set

$$\mathrm{Sel}(A^t)_{m^n} = \mathrm{Ker}\left[H^1(K,A^t_{m^n}) \to \bigoplus_{\text{all } v} H^1(K_v,A^t)\right]$$

and

$$T_m \,\mathrm{Sel}(A^t) = \varprojlim_n \mathrm{Sel}(A^t)_{m^n}.$$

Then the following holds[1]:

Main Theorem. *For any positive integer m, there exists a natural exact sequence of compact groups*

$$0 \leftarrow Ш(A)(m)^* \leftarrow H^1(K,A)(m)^* \leftarrow \prod_{\text{all } v} H^0(K_v,A^t)^{\frown} \leftarrow T_m\mathrm{Sel}(A^t) \leftarrow 0.$$

It should be noted that a similar statement holds true if above the henselizations of K are replaced by its completions. See [9, Remark I.3.10, p.58].

This paper grew out of questions posed to the authors by B.Poonen, in connection with the forthcoming paper [10]. We expect that the above theorem will be useful in [op.cit.].

Acknowledgements

K.-S.T. thanks J.Milne for the helpful outline [8] which led to a complete proof of the p-primary part of the Main Theorem of this paper under the assumption that $Ш(A)$ is finite. C.G.-A. thanks B.Poonen for suggesting Proposition 3.3 below and, more generally, for suggesting that the above finiteness assumption can be dispensed

[1] To see why the exact sequence of the theorem extends the Cassels-Tate dual exact sequence recalled above, see exact sequence (6) below and note that $T_m\,Ш(A) = T_m(Ш(A)_{m\text{-div}})$ vanishes if $Ш(A)_{m\text{-div}} = 0$.

with in the relevant proofs of [9] if $A(K)^\frown$ is replaced with $T_m \operatorname{Sel}(A)$ throughout[2].

2. Settings and notations

Let K be a global field and let A be an abelian variety over K. In the function field case, we let p denote the characteristic of K. All cohomology groups below are either Galois cohomology groups or flat cohomology groups. For any non-archimedean prime v of K, K_v will denote the field of fractions of the henselization of the ring of v-integers of K. If v is an archimedean prime, K_v will denote the completion of K at v, and we will write $H^0(K_v, A)$ for the quotient of $A(K_v)$ by its identity component. Note that, for any prime v of K, the group $H^1(K_v, A)$ is canonically isomorphic to $H^1(\widehat{K}_v, A)$, where $\widehat{K}_v$ denotes the completion of K at v. See [9, Remark I.3.10(ii), p.58]. Now let X denote either the spectrum of the ring of integers of K (in the number field case) or the unique smooth complete curve over the field of constants of K with function field K (in the function field case). In what follows, U denotes a nonempty open subset of X such that A has good reduction over U. When N is a quasi-finite flat group scheme on U, we endow $H^r(U, N)$ with the discrete topology. Now let m and n be arbitrary positive integers, and let M be an abelian topological group. We will write M/m^n for $M/m^nM = M \otimes_{\mathbb{Z}} \mathbb{Z}/m^n$ and $M^\frown$ for the m-adic completion $\varprojlim_n M/m^n$ of M. Further, we set $\mathbb{Z}_m = \prod_{\ell | m} \mathbb{Z}_\ell$, $\mathbb{Q}_m = \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Q}$ and define $M^* = \operatorname{Hom}_{\text{cts}}(M, \mathbb{Q}_m/\mathbb{Z}_m)$. Finally, the m-primary component of a torsion group M will be denoted by $M(m)$.

3. Proof of the Main Theorem

Both A and its dual variety A^t extend to abelian schemes $\mathcal{A}$ and $\mathcal{A}^t$ over U (see [2, Ch.1, §1.4.3]). By [5, VIII.7.1(b)], the canonical Poincaré biextension of (A^t, A) by $\mathbb{G}_m$ extends to a biextension over U of $(\mathcal{A}^t, \mathcal{A})$ by $\mathbb{G}_m$. Further, by [op.cit., VII.3.6.5], (the isomorphism class of) this biextension corresponds to a map $\mathcal{A}^t \otimes^{\mathbf{L}} \mathcal{A} \to \mathbb{G}_m[1]$ in the derived category of the category of smooth sheaves on U. This map in turn induces (see [9, p.283]) a canonical pairing $H^1(U, \mathcal{A}^t) \times H^1_{\mathrm{c}}(U, \mathcal{A}) \to H^3_{\mathrm{c}}(U, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$, where the $H^r_{\mathrm{c}}(U, \mathcal{A})$ are

---

[2] After this paper was completed, we learned that the existence of a natural duality between Б$(A)(m)$ and $T_m \operatorname{Sel}(A^t)$ had already been observed by J.W.S.Cassels in the case of elliptic curves over number fields. See [3, p.153]. Therefore, the Main Theorem of this paper may be regarded as a natural generalization of Cassels' result.

the cohomology groups with compact support of the sheaf $\mathcal{A}$ defined in [9, p.271].

Remark 3.1. The smoothness of $\mathcal{A}$ implies that the groups $H^r(U,\mathcal{A})$ and $H^r_{\mathrm{c}}(U,\mathcal{A})$ agree with the analogous groups defined for the étale topology. See [9, Proposition III.0.4(d), p.272].

For any positive integer m and any $n\geq 1$, the above pairing induces a pairing

$$H^1(U,\mathcal{A}^t_{m^n})\times H^1_{\mathrm{c}}(U,\mathcal{A})/m^n\to\mathbb{Q}/\mathbb{Z}. \tag{1}$$

On the other hand, the map $\mathcal{A}^t\otimes^{\mathbf{L}}\mathcal{A}\to\mathbb{G}_m[1]$ canonically defines a map $\mathcal{A}^t_{m^n}\times\mathcal{A}_{m^n}\to\mathbb{G}_m$, which induces a pairing

$$H^1(U,\mathcal{A}^t_{m^n})\times H^2_{\mathrm{c}}(U,\mathcal{A}_{m^n})\to\mathbb{Q}/\mathbb{Z}. \tag{2}$$

The preceding pairing induces an isomorphism

$$H^2_{\mathrm{c}}(U,\mathcal{A}_{m^n})\xrightarrow{\sim}H^1(U,\mathcal{A}^t_{m^n})^*. \tag{3}$$

See [9, Corollary II.3.3, p.217] for the case where m prime to p, and [op.cit., Theorem III.8.2, p.361] for the case where m is divisible by p. The pairings (1) and (2) are compatible, in the sense that the following diagram commutes:

$$\begin{tikzcd} H^1(U,\mathcal{A}^t_{m^n})\times H^1_{\mathrm{c}}(U,\mathcal{A})/m^n \arrow[r] \arrow[d, "\mathrm{id}\times\partial"] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\ H^1(U,\mathcal{A}^t_{m^n})\times H^2_{\mathrm{c}}(U,\mathcal{A}_{m^n}) \arrow[r] & \mathbb{Q}/\mathbb{Z}, \end{tikzcd} \tag{4}$$

where $\partial\colon H^1_{\mathrm{c}}(U,\mathcal{A})/m^n\hookrightarrow H^2_{\mathrm{c}}(U,\mathcal{A}_{m^n})$ is induced by the connecting homomorphism $H^1_{\mathrm{c}}(U,\mathcal{A})\to H^2_{\mathrm{c}}(U,\mathcal{A}_{m^n})$ coming from the exact sequence

$$0\to\mathcal{A}_{m^n}\to\mathcal{A}\overset{m^n}{\to}\mathcal{A}\to 0.$$

Now define

$$\mathrm{Sel}(A^t)_{m^n}=\mathrm{Ker}\left[H^1(K,A^t_{m^n})\to\bigoplus_{\text{all } v}H^1(K_v,A^t)\right]$$

and

$$T_m\,\mathrm{Sel}(A^t)=\varprojlim_n\mathrm{Sel}(A^t)_{m^n}.$$

By the proof of [9, Proposition I.6.4, p.92], there exists an exact sequence

$$0\to A^t(K)/m^n\to\mathrm{Sel}(A^t)_{m^n}\to\text{Ш}(A^t)_{m^n}\to 0. \tag{5}$$

Taking inverse limits, we obtain an exact sequence

$$0 \to A^t(K)\widehat{\ } \to T_m \operatorname{Sel}(A^t) \to T_m \text{Ш}(A^t) \to 0. \tag{6}$$

See [1, Proposition 10.2, p.104]. Now define[3]

$$D^1(U, \mathcal{A}^t_{m^n}) = \operatorname{Ker}\left[H^1(U, \mathcal{A}^t_{m^n}) \to \prod_{v \notin U} H^1(K_v, A^t)\right]$$

and

$$\begin{aligned} D^1(U, \mathcal{A}^t) &= \operatorname{Im}\left[\, H^1_{\mathrm{c}}(U, \mathcal{A}^t) \to H^1(U, \mathcal{A}^t)\,\right] \\ &= \operatorname{Ker}\left[\, H^1(U, \mathcal{A}^t) \to \prod_{v \notin U} H^1(K_v, A^t)\,\right]. \end{aligned}$$

Note that the pairing $H^1(U, \mathcal{A}^t) \times H^1_{\mathrm{c}}(U, \mathcal{A}) \to \mathbb{Q}/\mathbb{Z}$ induces a pairing $D^1(U, \mathcal{A}^t) \times D^1(U, \mathcal{A}) \to \mathbb{Q}/\mathbb{Z}$.

By [9, Proposition III.0.4(a), p.271] and the right-exactness of the tensor product functor, there exists a natural exact sequence

$$\bigoplus_{v \notin U} H^0(K_v, A)/m^n \to H^1_{\mathrm{c}}(U, \mathcal{A})/m^n \to D^1(U, \mathcal{A})/m^n \to 0. \tag{7}$$

Lemma 3.2. *The map* $H^1(U, \mathcal{A}^t_{m^n}) \hookrightarrow H^1(K, A^t_{m^n})$ *induces an isomorphism*

$$D^1(U, \mathcal{A}^t_{m^n}) \simeq \operatorname{Sel}(A^t)_{m^n}.$$

Proof. By [9, Lemma II.5.5, p.246] and Remark 3.1 above, the map $H^1(U, \mathcal{A}^t) \hookrightarrow H^1(K, A^t)$ induces an isomorphism

$$D^1(U, \mathcal{A}^t)_{m^n} \simeq \text{Ш}(A^t)_{m^n}.$$

Now $H^1(U, \mathcal{A}^t_{m^n}) \to \prod_{v \in U} H^1(K_v, A^t)$ factors through $H^1(U, \mathcal{A}^t) \to \prod_{v \in U} H^1(K_v, A^t)$, which is the zero map (see [9, (5.5.1), p.247] and Remark 3.1 above). Consequently, $H^1(U, \mathcal{A}^t_{m^n}) \hookrightarrow H^1(K, A^t_{m^n})$ maps $D^1(U, \mathcal{A}^t_{m^n})$ into $\operatorname{Sel}(A^t)_{m^n}$. To prove surjectivity, we consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}^t(U)/m^n\mathcal{A}^t(U) \arrow[r] \arrow[d] & H^1(U, \mathcal{A}^t_{m^n}) \arrow[r] \arrow[d] & H^1(U, \mathcal{A}^t)_{m^n} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A^t(K)/m^nA^t(K) \arrow[r] & H^1(K, A^t_{m^n}) \arrow[r] & H^1(K, A^t)_{m^n} \arrow[r] & 0. \end{tikzcd}$$

Note that the properness of $\mathcal{A}^t$ over U implies that the left-hand vertical map in the above diagram is an isomorphism (see [op.cit., p.242]). Now let $c \in \operatorname{Sel}(A^t)_{m^n}$, write c' for its image in $\text{Ш}(A^t)_{m^n}$ under the map in (5) and let $\xi' \in D^1(U, \mathcal{A}^t)_{m^n} \subset H^1(U, \mathcal{A}^t)_{m^n}$ be the pullback of

[3] In these definitions, the products extend over all primes of K, including the archimedean primes, not in U.

c' under the isomorphism $D^1(U,\mathcal{A}^t)_{m^n} \simeq Ш(A^t)_{m^n}$ recalled above. Then the fact that the left-hand vertical map in the above diagram is an isomorphism implies that ξ' can be pulled back to a class $\xi \in H^1(U,\mathcal{A}^t_{m^n})$ which maps down to c. Clearly $\xi \in D^1(U,\mathcal{A}^t_{m^n})$, and this completes the proof. $\square$

The following proposition generalizes [9, Theorem II.5.2(c), p.244].

Proposition 3.3. *There exists a canonical isomorphism*

$$(\,T_m\,\mathrm{Sel}(A^t))^* \xrightarrow{\sim} H^2_{\mathrm{c}}(U,\mathcal{A})(m).$$

Proof. There exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{c}}(U,\mathcal{A})/m^n \arrow[r] \arrow[dr, "c"] & H^2_{\mathrm{c}}(U,\mathcal{A}_{m^n}) \arrow[r] \arrow[d, "\simeq"] & H^2_{\mathrm{c}}(U,\mathcal{A})_{m^n} \arrow[r] & 0 \\ & & H^1(U,\mathcal{A}^t_{m^n})^*, & & \end{tikzcd}$$

where the vertical map is the isomorphism (3). Clearly, the above diagram induces an isomorphism $\mathrm{Coker}\, c \simeq H^2_{\mathrm{c}}(U,\mathcal{A})_{m^n}$. On the other hand, there exists a natural exact commutative[4] diagram

$$\begin{tikzcd} \bigoplus_{v\notin U} H^0(K_v,A)/m^n \arrow[r] \arrow[d, "\simeq"] & H^1_{\mathrm{c}}(U,\mathcal{A})/m^n \arrow[r] \arrow[d, "c"] & D^1(U,\mathcal{A})/m^n \arrow[r] \arrow[d, "\psi"] & 0 \\ \bigoplus_{v\notin U} H^1(K_v,A^t)^*_{m^n} \arrow[r] & H^1(U,\mathcal{A}^t_{m^n})^* \arrow[r] & D^1(U,\mathcal{A}^t_{m^n})^* \arrow[r] & 0, \end{tikzcd}$$

where the top row is (7), the right-hand vertical map ψ is the composite of the natural map $D^1(U,\mathcal{A})/m^n \to D^1(U,\mathcal{A}^t)^*_{m^n}$ induced by the pairing $D^1(U,\mathcal{A}^t)\times D^1(U,\mathcal{A}) \to \mathbb{Q}/\mathbb{Z}$ and the natural map $D^1(U,\mathcal{A}^t)^*_{m^n} \to D^1(U,\mathcal{A}^t_{m^n})^*$, and the left-hand vertical map is induced by the canonical Poincaré biextensions of (A^t,A) by $\mathbb{G}_m$ over K_v for each $v \notin U$. That the latter map is an isomorphism follows from [9, Remarks I.3.5 and I.3.7, pp.53 and 56, and Theorem III.7.8, p.354] and the fact that the pairings defined in [loc.cit.] are compatible with the pairing induced by the canonical Poincaré biextension (see [4, Appendix]). The above diagram and the identification $\mathrm{Coker}\, c = H^2_{\mathrm{c}}(U,\mathcal{A})_{m^n}$ yield an exact sequence

$$D^1(U,\mathcal{A})/m^n \to D^1(U,\mathcal{A}^t_{m^n})^* \to H^2_{\mathrm{c}}(U,\mathcal{A})_{m^n} \to 0$$

[4]The commutativity of this diagram follows from that of diagram (4).

Taking direct limits, we obtain an exact sequence

$$D^1(U,\mathcal{A})\otimes\mathbb{Q}_m/\mathbb{Z}_m\to\big(\varprojlim D^1(U,\mathcal{A}^t_{m^n})\big)^*\to H^2_{\rm c}(U,\mathcal{A})(m)\to 0$$

But $D^1(U,\mathcal{A})\otimes\mathbb{Q}_m/\mathbb{Z}_m=0$ since $D^1(U,\mathcal{A})$ is torsion and $\mathbb{Q}_m/\mathbb{Z}_m$ is divisible. Now lemma 3.2 completes the proof. □

By Remark 3.1 and [9, proof of Lemma II.5.5, p.247, and Proposition II.2.3, p. 203], there exist exact sequences[5]

$$H^1(U,\mathcal{A})\xrightarrow{c_U}\bigoplus_{v\notin U}H^1(K_v,A)\to H^2_{\rm c}(U,\mathcal{A})$$

and

$$0\to H^1(U,\mathcal{A})\xrightarrow{i_U}H^1(K,A)\xrightarrow{\lambda_U}\bigoplus_{v\in U}H^1(K_v,A),$$

where c_U and λ_U are natural localization maps and i_U is induced by the inclusion Spec$K\hookrightarrow U$. If $U\subset V$ is an inclusion of nonempty open subsets of X, then there exists a natural commutative diagram

$$\begin{tikzcd}
H^1(V,\mathcal{A}) \arrow[r, "c_V"] \arrow[d] & \bigoplus_{v\notin V} H^1(K_v,A) \arrow[d] \\
H^1(U,\mathcal{A}) \arrow[r, "c_U"] & \bigoplus_{v\notin U} H^1(K_v,A).
\end{tikzcd}$$

Define

$$Б(A)_U=\operatorname{coker}\left[c_U\colon H^1(U,\mathcal{A})\to\bigoplus_{v\notin U}H^1(K_v,A)\right],$$

which we regard as a subgroup of $H^2_{\rm c}(U,\mathcal{A})$. The preceding diagram shows that an inclusion $U\subset V$ of nonempty open subsets of X induces a map $Б(A)_V\to Б(A)_U$. Define

$$Б(A)=\varinjlim Б(A)_U=\operatorname{coker}\left[H^1(K,A)\to\bigoplus_{\text{all }v}H^1(K_v,A)\right],$$

where the limit is taken over the directed family of all nonempty open subsets U of X such that A has good reduction over U, ordered by

[5]In the second exact sequence, "$v\in U$" is shorthand for "v is a closed point of U".

$V \le U$ if and only if $U \subset V$. For each U as above and every $n \ge 1$, there exists an exact sequence

$$\bigoplus_{v \notin U} H^1(K_v, A)_{m^n} \to (Б(A)_U)_{m^n} \to (\operatorname{Im} c_{_U})/m^n.$$

Since $\operatorname{Im} c_{_U}$ is torsion, we conclude that there exists a surjection

$$\bigoplus_{v \notin U} H^1(K_v, A)(m) \twoheadrightarrow Б(A)_U(m) \tag{8}$$

On the other hand, by the proof of [9, Corollary I.6.23(b), p.111], there exists a natural injection $T_m \operatorname{Sel}(A^t) \hookrightarrow \prod_{\text{all } v} H^0(K_v, A^t)^{\wedge}$ and hence a surjection

$$\bigoplus_{\text{all } v} (H^0(K_v, A^t)^{\wedge})^* \to (T_m \operatorname{Sel}(A^t))^*$$

Further, as noted in the proof of Proposition 3.3, the canonical Poincaré biextensions induce an isomorphism

$$\bigoplus_{\text{all } v} (H^0(K_v, A^t)^{\wedge})^* \simeq \bigoplus_{\text{all } v} H^1(K_v, A)(m),$$

whence there exists a surjection

$$\bigoplus_{\text{all } v} H^1(K_v, A)(m) \twoheadrightarrow (T_m \operatorname{Sel}(A^t))^*. \tag{9}$$

The maps (8) and (9) fit into a commutative diagram

$$\begin{tikzcd}
\bigoplus_{\text{all } v} H^1(K_v, A)(m) \arrow[r, two heads, "(9)"] & (T_m \operatorname{Sel}(A^t))^* \arrow[r, "\sim"] & H^2_{\mathrm{c}}(U, \mathcal{A})(m) \\
\bigoplus_{v \notin U} H^1(K_v, A)(m) \arrow[u, hook] \arrow[r, two heads, "(8)"] & Б(A)_U(m), \arrow[u, hook, dashed] \arrow[ur, hook] &
\end{tikzcd}$$

where the isomorphism on the top row exists by Proposition 3.3. Taking the direct limit over U in the above diagram, we conclude that there exists an isomorphism

$$Б(A)(m) \xrightarrow{\sim} (T_m \operatorname{Sel}(A^t))^*,$$

as desired.

Remark 3.4. Recently [7, Theorem 1.2], the Cassels-Tate dual exact sequence has been extended to 1-motives M over number fields under the assumption that the Tate-Shafarevich group of M is finite. Now, using [6, Remark 5.10], it should not be difficult to extend this result

to global function fields, provided the p-primary components of the groups involved are ignored, where p denotes the characteristic of K. In this paper we have removed the latter restriction when M is an abelian variety, but the problem remains for general 1-motives M.

References

[1] Atiyah, M. and MacDonald, I.: *Introduction to Commutative Algebra.* Addison-Wesley, Reading, MA., 1969.

[2] Bosch, S., Lütkebohmert, W. and Raynaud, M.: *Néron Models.* Springer Verlag, Berlin 1989.

[3] Cassels, J.W.S: *Arithmetic on curves of genus 1. VII. The dual exact sequence.* J. Reine Angew. Math. **216**, no. 1, pp. 150-158 (1964).

[4] González-Avilés, C.D.: *Brauer groups and Tate-Shafarevich groups.* J. Math. Sciences, Univ. Tokyo **10**, no. 2 pp. 391-419 (2003).

[5] Grothendieck, A.: *Groupes de Monodromie en Géométrie Algébrique I*, Séminaire de Géométrie Algébrique du Bois Marie 1967-69 (SGA 7 I). Lecture Notes in Math., vol. **288**, Springer, Heidelberg, 1972.

[6] Harari, D. and Szamuely, T.: *Arithmetic duality theorems for 1-motives.* J. reine angew. Math. **578**, pp. 93-128 (2005).

[7] Harari, D. and Szamuely, T.: *On the arithmetic of 1-motives.* In preparation. Available from http://www.renyi.hu/∼szamuely.

[8] Milne, J.S.: *Letter to K.-S.Tan,* March 19th, 1991.

[9] Milne, J.S.: *Arithmetic Duality Theorems.* Persp. in Math., vol. 1. Academic Press Inc., Orlando 1986.

[10] Poonen, B. and Voloch, F.: *The Brauer-Manin obstruction for subvarieties of abelian varieties over function fields.* In preparation.

[11] Tate, J.: *Duality theorems in Galois cohomology over number fields.* Proceedings of the International Congress of Mathematicians, Stockholm, 1962, pp. 288-295.

Departamento de Matemáticas, Universidad Andrés Bello, Chile
E-mail address: `cristiangonzalez@unab.cl`

Department of Mathematics, National Taiwan University, Taiwan
E-mail address: `tan@math.ntu.tw`